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Abstract

In this paper, we establish some local stability estimate for a problem of determining a domain D appearing in the heat equation: $\frac{\partial u}{\partial t} - \Delta u + \chi_D\, u = 0$ from a Neumann additional data on a part of the lateral boundary. This estimate is obtained by taking advantage of some shape optimization tools.

Key words: Inverse problem ; Heat equation ; Local stability estimate ; Shape optimization

1 Introduction

Let Ω be an open bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and T be an arbitrary positive constant. Let $Q_T = \Omega \times (0,T)$, $\Sigma_0 = \Omega \times \{0\}$ and $\Sigma_T = \partial\Omega \times (0,T)$. We consider the following initial boundary value problem (IBVP)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \chi_D\, u = 0 & \text{in } Q_T \\ u = u_0 & \text{on } \Sigma_0 \\ u = f & \text{on } \Sigma_T, \end{cases} \tag{1}$$

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submitted

where χ_D denotes the characteristic function of a subdomain D of Ω, and u_0 and f are given functions defined respectively on Ω and Σ_T.

The IBVP (1) describes the heat conduction procedure in a given medium Ω and $\chi_D u(x,t)$ represents the discontinuous heat source. In the present paper we are concerned with the inverse problem consisting in the determination of heat conduction properties of the medium from additional information about the solution. Mathematically speaking, we discuss the following inverse problem: *Determine D in the IBVP (1) from the additional data on the flux* $\frac{\partial u(D)}{\partial \nu}|_{\Gamma_T}$, where $\Gamma_T = \Gamma \times (0,T)$, Γ is a subset of $\partial\Omega$ and $u(D)$ is the solution of IBVP (1) corresponding to D.
Here and henceforth $\frac{\partial}{\partial \nu}$ denotes the derivative with respect to the outward normal to $\partial\Omega$.

In recent years, there are several works devoted to the study of uniqueness and stability results for the inverse problem associated to steady state case governed by elliptic equations (see for instance [1,3,5,6,9,11] and the references therein). The main purpose of this paper is to extend the strategy from [5,6] to the parabolic IBVP (1). More precisely we examine the local stability in the inverse heat problem. That is to establish the following estimate:

$$\operatorname{meas}(D_V \Delta D) \leq C \left\| \left(\frac{\partial u(D_V)}{\partial \nu} - \frac{\partial u(D)}{\partial \nu} \right) |_{\Gamma_T} \right\|_{\mathcal{X}}, \tag{2}$$

where V is an admissible vector field, $D_V = (I+V)(D)$, I denotes the identity matrix of $\mathbb{R}^N$ and $\mathcal{X}$ is a suitable Sobolev space. The proof of this estimate relies on the Gâteaux-differentiability of the mapping: $V \to \frac{\partial u(D_V)}{\partial \nu}$ at $V = 0$ and the injectivity of its Gâteaux derivative, by taking advantage of some shape optimization tools.

Before closing this introduction, we notice that a problem of determining the diffusivity of a parabolic equation arising in hydrology, where the diffusivities are of the form $a(x) = A + K\chi_D(x)$ has been considered previously by Bellout [2] where entirely different methods were used. The author obtains a (weak) local stability result. In [4] Cannon and Pérez-Esteva obtain a logarithmic stability estimate for a problem of finding a region D, where D have a symmetic property, in the 3D heat equation: $\frac{\partial u}{\partial t} - \Delta u = f(t)\chi_D(x)$. They use a fundamental solution representation for the solution of this equation in order to derive the stability estimate. More recently, Hettlich and Rundell [10] considered a 2D heat equation of the form $\frac{\partial u}{\partial t} - \Delta u = \chi_D(x)$, where D is an unknown subdomain of a disc. They establish the uniqueness in determining the subdomain D from the measures of the solution at two different points on the boundary.

2 Position of the problem and statement of the main result

Throughout this paper, we assume that Ω is of class $\mathcal{C}^2$ and $D \subset\subset \Omega$ is of class $\mathcal{C}^1$. We suppose also that $f \in \mathcal{C}^{2,1}(\Sigma_T)$ and that $u_0 \in \mathcal{C}^1(\overline{\Omega})$. In this case, it is known (see, for example [8] or [12]) that the unique solution of IBVP (1), $u = u(D)$, associated to fixed D belongs to $H^{2,1}(Q_T) \cap \mathcal{C}(\overline{Q}_T)$, where $H^{r,s}(Q_T) := L^2(0,T; H^r(\Omega)) \cap H^s(0,T; L^2(\Omega))$, (for $r, s \in \mathbb{R}^+$).
For the study of the stability result, we will use shape optimization techniques which are based on the computation of the shape derivative of u with respect to D, in the direction of a vector field V. For this, we fixe Ω_0 an open subset of Ω with smooth boundary, such that $D \subset \overline{\Omega}_0 \subset\subset \Omega$ and we introduce the closed subspace of vector fields:

$$X = \{V \in \mathcal{C}^{1,b}(\mathbb{R}^N; \mathbb{R}^N) \ / \ \text{supp } V \subset \overline{\Omega}_0\}$$

where $\mathcal{C}^{1,b}(\mathbb{R}^N; \mathbb{R}^N)$ is the Banach space of vector fields in $\mathcal{C}^1(\mathbb{R}^N; \mathbb{R}^N)$ which are bounded and having bounded first derivatives. We define the quotient Banach space $Y = X/\mathcal{F}$, where $\mathcal{F} = \{V \in X \ / \ V \cdot n = 0\}$, where Y is equipped with the usual quotient norm, denoted by $\|\cdot\|_Y$, and n is the outward normal to ∂D. Next, we choose $\mathcal{U}$ a neighborhood of 0 in Y in such way that $D_V = (I+V)D$ is contained in $\overline{\Omega}_0$, for all $V \in \mathcal{U}$, where I denotes the identity matrix of $\mathbb{R}^N$. The aim of this paper is to show a stability result of type (2), for all $V \in \mathcal{U}$. For this, the basic idea is to show that the operator

$$\Lambda \ : \ V \in \mathcal{U} \longrightarrow \frac{\partial u_V}{\partial \nu} \,|_{\Gamma_T} \in \mathcal{X}$$

is Gâteaux-differentiable at $V = 0$ and its derivative is one to one, where $\mathcal{X}$ is an appropriate space and $u_V = u(D_V)$ is the solution of IBVP (1) with D_V in place of D. In order to state our main result, we need to introduce the following Sobolev space $H^{r,s}(\Gamma_T) := L^2(0,T; H^r(\Gamma)) \cap H^s(0,T; L^2(\Gamma))$ (for all $s,\ r \in \mathbb{R}^+$) equiped with its natural norm and for $r,\ s \leq 0$ we define by duality $H^{r,s}(\Gamma_T) := (H^{-r,-s}(\Gamma_T))'$.

Theorem 1. *Let us assume*

(H1) $\quad \Gamma$ *is a closed subset of* $\partial\Omega$ *with nonempty interior.*

(H2) $\quad f$ *is non negative and non identically null on* Σ_T,.

(H3) $\quad u_0 \geq 0$ *and* $\text{supp}\, u_0 \subset \Omega \setminus \overline{\Omega}_0$.

Then Λ *has the following properties* (**i**) Λ *is Gâteaux-differentiable at* $V = 0$ *and its derivative at* $V = 0$ *is given by*

$$\Lambda'(0)(V) := \lim_{s \longrightarrow 0} \frac{\Lambda(sV) - \Lambda(0)}{s} = \frac{\partial u'(V)}{\partial \nu} \in H^{-\frac{1}{2},-\frac{1}{4}}(\Gamma_T),$$

where $u'(V)$ *is the shape derivative of* u *in the direction* V *defined in (6).*
(**ii**) *Ker* $\Lambda'(0) = \{0_Y\}$.

3 Proof of Theorem 1 and local stability result

We first begin by showing that the restriction of the normal derivative of u_V on Γ_T is contained in the space $H^{-\frac{1}{2},-\frac{1}{4}}(\Gamma_T)$. This is stated in the following lemma.

Lemma 1. *The trace operator*

$$\begin{aligned} \gamma_1 \ : \ \mathcal{V}(W_T, \frac{\partial}{\partial t} - \Delta) \ &\longrightarrow \ H^{-\frac{1}{2},-\frac{1}{4}}(\Gamma_T) \\ \upsilon \ &\longrightarrow \ \frac{\partial \upsilon}{\partial \nu} \end{aligned}$$

is continuous, where

$$\mathcal{V}(W_T, \frac{\partial}{\partial t} - \Delta) = \{\upsilon \in L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega)) \ / \ \frac{\partial \upsilon}{\partial t} - \Delta \upsilon \in L^2(W_T)\}$$

and $W_T = \Omega \setminus \overline{\Omega}_0 \times (0,T)$

Proof. From Costabel [7], the trace operator γ_1 is continuous from $H^{1,\frac{1}{2}}(W_T, \frac{\partial}{\partial t} - \Delta) = \{\upsilon \in H^{1,\frac{1}{2}}(W_T) \ / \ \frac{\partial \upsilon}{\partial t} - \Delta \upsilon \in L^2(W_T)\}$ to $H^{-\frac{1}{2},-\frac{1}{4}}(\Gamma_T)$. Accordingly, the proof of the lemma follows from the continuous imbedding of $L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega))$ in $L^2(0,T; H^1(\Omega \setminus \overline{\Omega}_0)) \cap H^1(0,T; H^{-1}(\Omega \setminus \overline{\Omega}_0))$ and the density result of $L^2(0,T; H^1(\Omega \setminus \overline{\Omega}_0)) \cap H^1(0,T; H^{-1}(\Omega \setminus \overline{\Omega}_0))$ in $H^{1,\frac{1}{2}}(W_T)$ (see for example [7]). □

Now, the proof of the theorem is based on the following steps. The first step consists in proving Gâteaux-differentiability of Λ at $V = 0$.

Proposition 1. Λ *is Gâteaux-differentiable at* 0 *and* $\Lambda'(0)(V) = \frac{\partial u'(V)}{\partial \nu}|_{\Gamma_T}$, *where $u'(V) \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ is the solution of the IBVP*

$$\begin{cases} \dfrac{\partial u'}{\partial t} - \Delta u' + \chi_D\, u' = \mu(V) & \text{in } Q_T \\ u' = 0 & \text{in } \Sigma_0 \\ u' = 0 & \text{on } \Sigma_T, \end{cases} \tag{3}$$

where $\mu(V) \in L^2(0,T;H^{-1}(\Omega))$ is defined by

$$\langle \mu(V), \psi \rangle = -\int_0^T \int_{\partial D} (V \cdot n)\, u\, \psi \qquad \text{for all } \psi \in L^2(0,T;H_0^1(\Omega)),$$

here n denotes the outward normal to ∂D and $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $L^2(0,T;H^{-1}(\Omega))$ and $L^2(0,T;H_0^1(\Omega))$.

Proof. Let $F_0 \in \mathcal{C}^{2,1}(\overline{Q}_T)$, be such that $F_0|_{\Sigma_T} = f$. We consider $\Psi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, such that

$$\begin{cases} \Psi = 1 \text{ on } \Xi(\partial\Omega) \\ \operatorname{supp} \Psi \subset \Omega \setminus \overline{\Omega}_0, \end{cases}$$

where $\Xi(\partial\Omega)$ is a neighborhood of Γ. We denote by F the function defined by $F = \Psi\, F_0$. Since $\chi_D\, F = 0$ on Q_T, it follows easily that $v = u - F$ is solution of the IBVP

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + \chi_D\, v = G & \text{in } Q_T \\ v = U_0 & \text{on } \Sigma_0 \\ v = 0 & \text{on } \Sigma_T, \end{cases} \tag{4}$$

where $U_0(x) = u_0(x) - F(x,0)$ and $G = \Delta F - \frac{\partial F}{\partial t}$. In addition, from Renardy [13], the unique solution v of (4) belongs to the space $L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$.

Now, we will compute the shape derivative of v with respect to D. For this, we fixe $V \in X$ and we choose a small interval $J_\varepsilon = (-\varepsilon, \varepsilon)$ in such way that $D_s = (I + sV)D \subset \overline{\Omega}_0$, for all $s \in J_\varepsilon$. Let us consider $v_s = u_s - F$ the solution of IBVP (4) with D_s in place of D. As in [14], we define the material derivative of v in the direction of the vector field V by

$$\dot{v} = \lim_{s \longrightarrow 0} \frac{v^s - v}{s},$$

where $v^s = v_s \circ T_s$, $T_s = T_s(V) = ((I + sV), I_1)$ and I_1 is the identity mapping of $\mathbb{R}$, and $v = v(D)$ is the solution of IBVP (4). In order to determine the material derivative $\dot{v}$ in the direction of a vector field V it should be remarked that v_s satisfy the following integral identity

$$
\begin{aligned}
-\int\!\!\int_{Q_T} v_s \frac{\partial \varphi}{\partial t}\, dx\, dt + \int\!\!\int_{Q_T} \nabla v_s \cdot \nabla \varphi\, dx\, dt + \int\!\!\int_{Q_T} \chi_{D_s}\, v_s\, \varphi\, dx\, dt \\
= \int\!\!\int_{Q_T} G\, \varphi\, dx\, dt + \int_{\Sigma_0} U_0\, \varphi\, dx,
\end{aligned}
$$

for all $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$, such that $\varphi(x,T) = 0$.
Using the change of variable $x = T_s(X)$. It can be shown that $v^s = v_s \circ T_s$ is solution of the following integral identity

$$
\begin{aligned}
-\int\!\!\int_{Q_T} \gamma(s)\, v^s \frac{\partial \varphi}{\partial t}\, dx\, dt + \int\!\!\int_{Q_T} A(s)\, \nabla v^s \cdot \nabla \varphi\, dx\, dt \int\!\!\int_{Q_T} \chi_D\, v^s\, \varphi\, dx\, dt \\
= \int\!\!\int_{Q_T} G^s\, \varphi\, dx\, dt + \int_{\Sigma_0} U_0^s\, \varphi\, dx
\end{aligned}
$$

for all $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$, such that $\varphi(x,T) = 0$. Here $\gamma(s) = \det(I + sDV)$, $A(s) = \gamma(s) DT_s^{-1}\, {}^t(DT_s^{-1})$, $U_0^s = \gamma(s)\, U_0 \circ T_s$ and $G^s = \gamma(s)\, G \circ T_s$, with $DV = (\partial_i V_j)_{1 \le i,j \le n}$, DT_s^{-1} denotes the inverse of the matrix DT_s and ${}^t(DT_s^{-1})$ denotes the transpose of the matrix DT_s^{-1}. We note that the small interval J_ε is also chosen such that $\gamma(s) \geq 0$, for all $s \in J_\varepsilon$. Arguing as in Sokolowski and Zolesio [14] and using the following relations

$$
\lim_{s \longrightarrow 0} \frac{\gamma(s) - 1}{s} = \operatorname{div}(V), \quad \lim_{s \longrightarrow 0} \frac{A(s) - I}{s} = A' = DV +^t (DV) - \operatorname{div}(V)\, I
$$

and

$$
\lim_{s \longrightarrow 0} \frac{U_0^s - U_0}{s} = \nabla U_0 \cdot V, \quad \lim_{s \longrightarrow 0} \frac{G^s - G}{s} = \nabla G \cdot V\,,
$$

we can prove that $\dot{v}$ exists in the space $L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ and satisfies the following IBVP

$$
\begin{cases}
\dfrac{\partial \dot{v}}{\partial t} - \Delta \dot{v} + \chi_D\, \dot{v} & \\
\quad = \operatorname{div}(G\, V) - \dfrac{\partial v}{\partial t} \operatorname{div}(V) + \operatorname{div}(A' \nabla v) - \chi_D\, v \operatorname{div}(V) & \text{in } Q_T \\
\dot{v} = \operatorname{div}(U_0\, V) & \text{on } \Sigma_0 \\
\dot{v} = 0 & \text{on } \Sigma_T.
\end{cases} \tag{5}
$$

Now, since supp $U_0 \subset \Omega/\overline{\Omega}_0$ and supp $V \subset \overline{\Omega}_0$, we have that $\dot{v} \mid_{\Sigma_0} = 0$. By using the following known formula

$$\operatorname{div}(A' \nabla v) = \Delta v \operatorname{div}(V) - \Delta(\nabla v \cdot V) + \nabla(\Delta v) \cdot V$$

and the fact that v', the shape derivative of v, is given by

$$v' = \dot{v} - \nabla v \cdot V, \tag{6}$$

we can show that v' is solution of the IBVP

$$\begin{cases} \dfrac{\partial v'}{\partial t} - \Delta v' + \chi_D \, v' = \mu(V) \text{ in } \; Q_T \\ v' = 0 & \text{on } \Sigma_0 \\ v' = 0 & \text{on } \Sigma_T, \end{cases} \tag{7}$$

where $\mu(V) = v \, \nabla \chi_D \cdot V \in L^2(0,T; H^{-1}(\Omega))$ is defined, for all $\psi \in L^2(0,T; H_0^1(\Omega))$, by

$$\begin{aligned} < \mu(V), \psi > &= < v \, \nabla \chi_D \cdot V, \psi > \\ &= - \int\int_{Q_T} \chi_D \operatorname{div}(v \, \psi \, V) \\ &= - \int_0^T \int_D \operatorname{div}(v \, \psi \, V). \end{aligned}$$

Since D is chosen enough regular, an application of the Green's formula leads to

$$< \mu(V), \psi > = - \int_0^T \int_{\partial D} v \, \psi \, (V \cdot n).$$

We note that from Renardy [13], the solution v' of (7) is in the space $L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega))$. Since we have $u^s = v^s + F^s$ (with $F^s = F \circ ((I + sV), I_1))$, then the shape derivative of u is given by $u' = v' + F'$. Following Sokolowski and Zolesio [14], we have that

$$F' = \lim_{s \longrightarrow 0} \frac{F^s - F}{s} - \nabla F \cdot V = 0$$

and consequently $u' = v' \in L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega))$ is solution of IBVP (7) with $\mu(V) = u \, \nabla \chi_D \cdot V$ (because $F \, \nabla \chi_D \cdot V = 0$).

Next, since $u_s = u(D_s)$ the solution of IBVP (1), with D_s in place of D, is in $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ and satisfies the following equation in $W_T = \Omega \setminus \overline{\Omega}_0 \times (0,T)$

$$\frac{\partial u_s}{\partial t} - \Delta u_s = 0 \quad \text{in} \quad W_T, \tag{8}$$

we have that $u_s \in \mathcal{V}(W_T, \frac{\partial}{\partial t} - \Delta)$ and consequently $u^s = u_s \circ T_s$ is also in $\mathcal{V}(W_T, \frac{\partial}{\partial t} - \Delta)$. From equation (3) we have that u' exists in the space $L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ and satisfies the equation

$$\frac{\partial u'}{\partial t} - \Delta u' = 0 \quad \text{on} \quad W_T. \tag{9}$$

Therefore, we have that u' exists in $\mathcal{V}(W_T, \frac{\partial}{\partial t} - \Delta)$. Hence, we have shown that the mapping

$$s \in J_\varepsilon = (-\varepsilon, \varepsilon) \longrightarrow u^s = u_s \circ T_s \in \mathcal{V}(W_T, \frac{\partial}{\partial t} - \Delta)$$

is differentiable at $s = 0$ and its derivative at $s = 0$ is equal to $u' \mid_{W_T}$. Moreover, from Lemma 1, we have that the trace operator

$$\omega \in \mathcal{V}(W_T, \frac{\partial}{\partial t} - \Delta) \longrightarrow \frac{\partial \omega}{\partial \nu}|_{\Gamma_T} \in H^{-\frac{1}{2},-\frac{1}{4}}(\Gamma_T)$$

is continuous. This imply that the operator

$$\Lambda \ : \ \theta \in \mathcal{U} \longrightarrow \frac{\partial u_\theta}{\partial \nu}|_{\Gamma_T} \in H^{-\frac{1}{2},-\frac{1}{4}}(\Gamma_T),$$

has directional derivative at 0 in the direction V given by

$$\Lambda'(0)(V) = \lim_{s \longrightarrow 0} \frac{\Lambda(sV) - \Lambda(0)}{s} = \frac{\partial u'(V)}{\partial \nu} \in H^{-\frac{1}{2},-\frac{1}{4}}(\Gamma_T).$$

Finally, to show that Λ is Gâteaux-differentiable at 0, it remains to prove that the operator

$$\Lambda'(0) \ : \ V \in Y \longrightarrow \frac{\partial u'(V)}{\partial \nu} \in H^{-\frac{1}{2},-\frac{1}{4}}(\Gamma_T)$$

is bounded. Indeed, let $V \in X$. On account of the continuity of the trace operator from $H^{1,\frac{1}{2}}(W_T, \frac{\partial}{\partial t} - \Delta)$ to $H^{-\frac{1}{2},-\frac{1}{4}}(\Gamma_T)$, due to Costabel [7] there exists $C_1 > 0$ such that,

$$
\begin{aligned}
\|\Lambda'(0)(V)\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Gamma_T)} &= \|\frac{\partial u'(V)}{\partial \nu}\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Gamma_T)} \\
&\leq C_1 \|u'(V)\|_{H^{1,\frac{1}{2}}(W_T)}.
\end{aligned}
$$

On the other hand, since $u' \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)) \subset L^2(0,T;H_0^1(\Omega\setminus\overline{\Omega}_0)) \cap H^1(0,T;H^{-1}(\Omega\setminus\overline{\Omega}_0))$ and $\frac{\partial u'}{\partial t} - \Delta u' = 0$, it follows from [7] that there exists two constants $C_2 > 0$ such that

$$
\|u'(V)\|_{H^{1,\frac{1}{2}}(W_T)} \leq C_2 \|u'(V)\|_{L^2(0,T;H^1(\Omega\setminus\overline{\Omega}_0))} \leq C_2 \|u'(V)\|_{L^2(0,T;H_0^1(\Omega))}.
$$

Using an a priori estimate for the solution of IBVP (3) (see Renardy [13]), we conclude that

$$
\|u'(V)\|_{L^2(0,T;H_0^1(\Omega))} \leq C_3 \|\mu(V)\|_{L^2(0,T;H^{-1}(\Omega))},
$$

where C_3 is a positive constant. By elementary calculation we obtain

$$
\begin{aligned}
\|\Lambda'(0)(V)\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Gamma_T)} &\leq C_4 \|V\cdot n\|_{L^\infty(\partial D)} \\
&\leq C_4 \|(V+W)\cdot n\|_{L^\infty(\partial D)}, \text{ for all } W \in \mathcal{F} \\
&\leq C_4 \|V\|_Y
\end{aligned}
$$

where C_4 is a positive constant independent of V. This ends the proof of the proposition. □

Now, to prove that $\Lambda'(0)$ is injective, we need the two following lemmas

Lemma 2. *Assume that f and u_0 satisfy the hypotheses (H2)-(H3). Then the solution of IBVP (1) satisfies $u > 0$ on $\partial D \times (0,T)$.*

Proof. Let $H_0^{2,1}(Q_T) := \{\varphi \in H^{2,1}(Q_T) \text{ such that } \varphi\,|_{\Sigma_T} = 0\}$. It is easily seen that u is solution of the following integral identity:

$$
\begin{aligned}
&\int_\Omega u(x,\tau)\,\varphi(x,\tau)\,dx - \int_{\Sigma_0} u_0\varphi\,dx \\
&\qquad - \int\int_{Q_\tau} u\,\frac{\partial \varphi}{\partial t}\,dx\,dt + \int\int_{Q_\tau} \nabla u\cdot\nabla\varphi\,dx\,dt + \int\int_{Q_\tau} \chi_D\,u\,\varphi\,dx\,dt = 0,
\end{aligned}
$$

for a.e. $\tau \in (0,T)$ and for all $\varphi \in H_0^{2,1}(Q_T)$. By Choosing $\varphi = -u_- \in H_0^{2,1}(Q_T)$ together with the relations

$$
u^+u^- = \nabla u^+\cdot\nabla u^- = u^+\frac{\partial u^-}{\partial t} = u^-\frac{\partial u^+}{\partial t} = 0, \quad \text{for a. e.}
$$

where $u_- := -\inf(u, 0)$ and $u_+ := \sup(u, 0)$, we obtain

$$\int_\Omega \frac{1}{2}(u^-(x,\tau))^2 dx + \int_{Q_\tau} |\nabla u^-|^2 dxdt + \int_{Q_\tau} \chi_D(\ u^-)^2 dxdt = 0.$$

It follows that $u^- = 0$ a.e. on Q_T and therefore $u = u^+ \geq 0$ on Q_T.

Now, we suppose by absurd that there exists $(x_0, t_0) \in \partial D \times (0, T)$ such that $u(x_0, t_0) = 0$. By using Harnack inequality for weak solution (see [15]), we can construct two small cubes R^+ and R^- contained in Q_T such that $(x_0, t_0) \in R^-$ and

$$\sup_{R^+} u \leq C \inf_{R^-} u$$

where C is a constant independent of u. Thus u is identically null on R^+. According to the uniqueness of continuation property we can conclude that $u = 0$ on Q_T. But this contradicts the fact that $f = u\mid_{\Sigma_T}$ is non identically null. $\square$

Lemma 3. *For all $V \in X$, if $\Lambda'(0)V = 0$ then $\mu(V) = 0$.*

Proof. Let V in X such that $\Lambda'(0)V = 0$. Then from Proposition 1, we have $\frac{\partial u'(V)}{\partial \nu}\mid_{\Gamma_T} = 0$. We define $u'_e := u'(V)\mid_{\Omega\setminus\overline{D}\times(0,T)}$. By using the fact that $\mu(V)$ is supported on $\partial D \times (0, T)$, we conclude that u'_e is solution of the problem

$$\begin{cases} \dfrac{\partial u'_e}{\partial t} - \Delta u'_e = 0 & \text{in } (\Omega \setminus \overline{D}) \times (0, T) \\ u'_e = 0 & \text{on } \Sigma_T, \\ \dfrac{\partial u'_e}{\partial \nu} = 0 & \text{on } \Gamma \times (0, T) \end{cases} \tag{10}$$

According to the uniqueness of the continuation for the heat equation, we deduce that $u'_e = 0$ on $(\Omega \setminus \overline{D}) \times (0, T)$.

Now, set $u'_i := u'(V)\mid_{D\times(0,T)}$. we verify that u'_i is solution of the following IBVP

$$\begin{cases} \dfrac{\partial u'_i}{\partial t} - \Delta u'_i + u'_i = 0 & \text{in } D \times (0, T) \\ u'_i(x, 0) = 0 & \text{on } D, \\ u'_i = 0 & \text{on } \partial D \times (0, T)\,. \end{cases} \tag{11}$$

It follows from the uniqueness of the solution that $u'_i = 0$ in $D \times (0,T)$ and therefore $u'(V) = 0$ in $\Omega \times (0,T)$. This leads to desired conclusion. $\square$

Proof of Theorem 1. The item (i) was established in Proposition 1. To prove (ii), let V be in Ker $\Lambda'(0)$. According to Lemma 1, it follows that $\mu(V) = 0$ and thus $V \cdot n\ u = 0$ on $\partial D \times (0,T)$. But from Lemma 2, we known that $u > 0$ on $\partial D \times (0,T)$. This leads to $V \in \mathcal{F}$, namely $V = 0_Y$. $\square$

As consequence of the theorem, we state the following local stability result:

Corollary 1. *Let assumptions $(H1) - (H3)$ hold. Then for all $V \in X \setminus \mathcal{F}$, there exists two constants $\varepsilon = \varepsilon(V)$ and $k = k(V)$ such that:*

$$\text{meas } (D_{sV} \Delta D) \leq k \| \frac{\partial u_{sV}}{\partial \nu} - \frac{\partial u}{\partial \nu} \|_{H^{-\frac{1}{2}, -\frac{1}{4}}(\Gamma_T)}, \qquad \text{for all } s \in]-\varepsilon, \varepsilon[,$$

where $u(sV)$ is the solution of IBVP (1) associated to D_{sV}.

Proof. The proof of this result follows from the fact that

$$\lim_{s \longrightarrow 0} \| \frac{\Lambda(sV) - \Lambda(0)}{s} \| = \|\Lambda'(0)(V)\| > 0 \quad \text{for all } V \in X \setminus \mathcal{F}$$

and the inequality (see for instance [6]):

$$\text{meas } (D_{sV} \Delta D) \leq k_1(V)|s|.$$

Acknowledgements

This work was carried out in the framework of the on going project "Inverse problems and shape optimization" when the first and the third authors was visiting Professors at Laboratoire de Mathématiques Jean Leray, UMR662, CNRS/Université de Nantes/École Centrale de Nantes.

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